Published by Institute of Physics Publishing for SISSA/ISAS

Received: *January 23, 2003*
Accepted: *February 5, 2003*

Quark and lepton masses and mixing angles from superstring constructions

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Abstract: We show that the observed structure of quark and lepton masses and mixing angles can arise entirely geometrically from superstring constructions, at the renormalizable level. The model we consider is a $\mathbb{Z}_3$ orbifold compactification of heterotic string with two Wilson lines, where three families of particles of $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, including Higgses, are automatically present. In orbifold models, Yukawa couplings can be calculated explicitly, and it is known that they get exponential suppression factors depending on the distance between the fixed points to which the fields are attached. We find that in the $\mathbb{Z}_3$ case, the quark and charged-lepton mass hierarchies can easily be obtained for reasonable values of the three moduli determining the radii of the compactified space, $T_i \sim 1$. For the neutrinos, due to the smallness of their Dirac masses, the required scale for the see-saw mechanism to give the correct masses is found to be within reach of the electroweak scale. Finally, we find that one of the small number of possibilities for quark and lepton mass matrices yields consistent results for the mixing angles and the weak CP violation phase. Although our scheme relies on the mixing between fields due to Fayet-Iliopoulos breaking, it is considerably more predictive than alternative models of flavour.

Keywords: Superstrings and Heterotic Strings, Quark Masses and SM Parameters, Supersymmetric Standard Model, Compactification and String Models.

http://jhep.sissa.it/archive/papers/jhep022003010/jhep022003010.pdf

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1. Introduction

The Higgs mechanism [1] is the crucial ingredient in the Standard Model required to explain electroweak symmetry breaking [2], and hence the masses of the $W^{\pm}$ and Z gauge bosons. As an added bonus, the vacuum expectation value (VEV) of the Higgs field generates fermion masses through Yukawa couplings. However, the Standard Model does not address the origin of these couplings and the peculiar hierarchies that are required to reproduce the observed structure of quark and lepton masses [3]:

$$\begin{array}{lll} m_u = 1.5 \text{ to } 4.5\,\text{MeV}\,, & m_c = 1.0 \text{ to } 1.4\,\text{GeV}\,, & m_t = 174.3 \pm 5.1\,\text{GeV}\,, \\ m_d = 5 \text{ to } 8.5\,\text{MeV}\,, & m_s = 80 \text{ to } 155\,\text{MeV}\,, & m_b = 4.0 \text{ to } 4.5\,\text{GeV}\,, \\ m_e = 0.51\,\text{MeV}\,, & m_\mu = 105.658\,\text{MeV}\,, & m_\tau = 1.777\,\text{GeV}\,. \end{array} \tag{1.1}$$

In the framework of the Standard Model these are initial parameters that are put in by hand. In addition, the Yukawa couplings have to have off diagonal elements, with the

Cabibbo–Kobayashi–Maskawa (CKM) weak coupling matrix [4] arising from the matrices that diagonalize the up- and down-quark mass matrices have the form [3]

$$V_{CKM} = \begin{pmatrix} 0.9741 \text{ to } 0.9756 & 0.219 \text{ to } 0.226 & 0.0025 \text{ to } 0.0048 \\ 0.219 \text{ to } 0.226 & 0.9732 \text{ to } 0.9748 & 0.038 \text{ to } 0.044 \\ 0.004 \text{ to } 0.014 & 0.037 \text{ to } 0.044 & 0.9990 \text{ to } 0.9993 \end{pmatrix}, \tag{1.2}$$

The understanding, within some proposed extension of the Standard Model, of the particle masses (1.1) and the elements of the CKM matrix (1.2), remains one of the most important goals in particle physics. In addition one would like to understand the mixings and hierarchies of neutrinos. The global analysis of solar, atmospheric and reactor data [5] indicates that they have masses given by

$$\Delta m^2_{21} \approx 2.4 \times 10^{-5} \text{ to } 2.4 \times 10^{-4}\,\mathrm{eV}^2\,, \qquad \Delta m^2_{32} \approx 1.4 \times 10^{-3} \text{ to } 6 \times 10^{-3}\,\mathrm{eV}^2\,, \tag{1.3}$$

and a Maki-Nakagawa-Sakata (MNS) weak coupling matrix [6] with the charged leptons [5]:

$$V_{MNS} = \begin{pmatrix} 0.73 \text{ to } 0.89 & 0.45 \text{ to } 0.66 & < 0.24 \\ 0.23 \text{ to } 0.66 & 0.24 \text{ to } 0.75 & 0.52 \text{ to } 0.87 \\ 0.06 \text{ to } 0.57 & 0.40 \text{ to } 0.82 & 0.48 \text{ to } 0.85 \end{pmatrix}. \tag{1.4}$$

Since string theory is the prime candidate for the fundamental theory of particle physics (from which the Standard Model might be derived as a low-energy limit), we think that it must be able, in principle, to tackle these questions directly. In this paper, we present a stringy explanation for all of the fermion masses and mixings which is intended to take us a further step in this direction. Our approach will be 'bottom-up', in the sense that we will not be presenting a completely explicit string construction, but will be asking if there is a particle assignment (to for example different twisted sectors) in a particular orbifold construction, that can explain the observed structure of masses.

String theory does provide some striking hints that a natural explanation for the structure of masses and mixings might be possible. Indeed it is well known that abelian $\mathbb{Z}_n$ orbifold compactifications [7, 8] of the Heterotic Superstring have a beautiful geometric mechanism to generate a mass hierarchy [9]–[12]. $\mathbb{Z}_n$ orbifolds have twisted fields which are attached to orbifold fixed points. Fields at different fixed points can communicate with each other only via world sheet instantons. The resulting renormalizable Yukawa couplings can be explicitly computed [9] and [13]–[16] and those between fields in twisted sectors get exponential suppression factors that depend on the distance between the fixed points to which the relevant fields are attached. These distances can be varied by giving different VEVs to the T-moduli associated with the size and shape of the orbifold. The question of hierarchies is then translated into the question of why the moduli fields take the values that they do. Although an explanation for this lies outside the realm of perturbative string theory, generally there are far less moduli than there are hierarchies to be explained, and the nett outcome of such a geometric approach is an impressively large set of mass predictions.[1]

[1]Note that throughout the paper we use the word "prediction" in the very specific sense that it is

A strictly geometric explanation of Yukawa hierarchies means that the Yukawa couplings must be entirely renormalizable, since non-renormalizable terms introduce a dependence on the VEVs of the fields entering in the non-renormalizable couplings. Purely renormalizable Yukawa couplings are preferable, because the arbitrariness of such VEVs inevitably means that predictivity is lost. Furthermore, higher-order operators, such as those induced by the Fayet-Iliopoulos (FI) breaking [17], are very model-dependent and introduce a high degree of uncertainty in the computation. In addition, as emphasized in ref. [18], their presence is not always allowed in string constructions. For example, in the $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ model of ref. [19], those few non-renormalizable couplings that were allowed by gauge invariance were forbidden by the 'stringy' selection rules.

In specific orbifold constructions however, entirely renormalizable couplings seem to be unable to explain the experimental data. Summarizing the analyses in refs. [11, 12], for prime orbifolds the space group selection rule and the need for a hierarchy forces the fermion mass matrices to be diagonal at the renormalizable level. Thus in these cases the CKM parameters must arise at the non-renormalizable level. For non-prime orbifolds the mass matrices can actually be non-diagonal at the renormalizable level, however a reasonable CKM matrix again requires non-renormalizable couplings. In both cases, non-renormalizable couplings have to account for the masses of the first generation (coming from off-diagonal elements in the mass matrices), but renormalizable couplings can still be responsible for the masses of the second and third generation. Under this assumption it was shown that, for a reasonable size and shape of the compactified space, the $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$-I, and possibly $\mathbb{Z}_7$ orbifolds can fit the physical quark and charged-lepton masses adequately, but that the rest of the $\mathbb{Z}_n$ orbifolds cannot. In all cases though, non-renormalizable terms were necessary to fit the physical masses.

How might we avoid having to use non-renormalizable terms? One hint lies in the fact that their apparent necessity in the $\mathbb{Z}_3$ orbifold was deduced assuming a minimal $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ scenario with a single generation of supersymmetric Higgses (H^u, H^d). This is the usual assumption in the context of the Minimal Supersymmetric Standard Model (MSSM). In addition, three Wilson lines were assumed. This implies that the 27 twisted sectors of the $\mathbb{Z}_3$ orbifold are different, and then it is possible in principle to assign a physical field to any fixed point. On the one hand, this is welcome since it allows one to play fully with suppression factors and hence to obtain a realistic fermion mass hierarchy, albeit with nonrenormalizable terms. On the other hand, it is problematic, since the existence of three families of $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ is not guaranteed in all cases. The latter is also true for all other orbifolds.

In contrast to the situation with three Wilson lines, $\mathbb{Z}_3$ orbifold models with two Wilson lines automatically have three families of everything, including Higgses. This is because, in addition to the overall factor of 3 coming from the right-moving part of the untwisted

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commonly used when analyzing fermion masses. That is one assumes values for a set of parameters (for example charges in the case of Froggatt-Nielsen models). Then *given* those values one asks how many variables (such as VEVs) are fixed by masses. Once the variables are all fixed, the remaining masses are "predictions". In the present case, the T moduli VEVs are variables to be fixed by some of the masses, but the assumption of say a $\mathbb{Z}_3$ orbifold on an orthogonal lattice can be considered as an initial assumption.

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matter, the twisted matter comes in 9 sets with 3 equivalent sectors in each one, since there are 27 fixed points. Consequently these compactifications are very interesting from a phenomenological point of view. Indeed, several models with two Wilson lines, have been constructed with $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^n$ observable gauge group and three families of particles [20]. In addition, using the FI mechanism [17] it is possible to break the original gauge group down to $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ [21, 19, 22, 23]. Given these interesting properties, the aim of the present paper is to analyze as systematically as possible the structure of Yukawa couplings in $\mathbb{Z}_3$ orbifold models with two Wilson lines, and to ascertain whether a purely renormalizable structure is possible.

To carry out the analysis, we first make the most natural assumption, which is that all three generations of supersymmetric Higgses remain light (H_i^u, H_i^d), $i = 1, 2, 3$.[2] Indeed, in all the models that have been constructed, extra doublets are generically present at low energies. This possibility also favours the unification of gauge couplings in Heterotic Superstring constructions [18]. Importantly for our analysis, having three families of Higgses introduces more Yukawa couplings. The FI breaking provides a second important ingredient in our analysis which appears at the renormalizable level. Namely, after the gauge breaking some physical particles appear combined with other states, and the Yukawa couplings are modified in a well controlled way. This, of course, introduces more flexibility in the computation of the mass matrices.

All of these factors allow us to achieve our goal of obtaining realistic quark and lepton mass matrices entirely at the *renormalizable* level, and we will show below that the observed structure in eqs. (1.1) and (1.2), can indeed be obtained. Despite the modifications due to three Higgs families and FI mixing, the model retains a large degree of predictivity. Indeed, if we assume one overall modulus we successfully predict two mixing angles and *four* masses (out of a total of three angles and nine quark and charged lepton masses).

$\mathbb{Z}_3$ orbifold models with two Wilson lines also allow us to attack the problem of neutrino masses and here we find some rather attractive features. It turns out that we require a see-saw mechanism to generate small enough neutrino masses. However, the Dirac mass matrices for neutrinos are already very small in these constructions because the Yukawa couplings get the same kind of hierarchical suppression that appears in the quarks and charged leptons couplings. For the examples we present in the text, we will find that inserting the measured neutrino masses (assuming that the mass-squared differences reflect the actual masses) then tells us that the see-saw scale has to be $\sim 10^4$ TeV. This generates an effective μ-term of order 500 TeV which is only a couple of orders of magnitude above the weak scale. We think that the *relative* closeness of the see-saw scale to the electroweak scale (compared to the usual see-saw mechanism) may be an important hint. First, it suggests that the three families of singlet responsible for the neutrino see-saw mechanism may get their VEVs when electroweak symmetry is broken. Second it suggests that the solution

[2] One might avoid the extra light Higgs generations if they became quite massive due to some kind of 'asymmetric' breaking. However, this is certainly not natural in these models. For example, one could generate a high mass for the Higgses through trilinear couplings involving a field which develops a VEV in order to cancel the FI D-term. However, given the structure of Yukawa couplings in these models (see section 2), all three families of Higgses would become massive.

to the so called μ-problem is that the Higgs fields couple to the same singlets as give a mass to the right handed neutrinos, possibly through slightly suppressed couplings. The maximal mixing and rather mild hierarchies of the neutrino system can also be achieved with no other modifications.

Of course, dangerous flavour-changing neutral currents (FCNCs) may appear when three generations of Higgses are present [24]. In general, the most stringent limit on flavour-changing processes comes from the small value of the $K_L - K_S$ mass difference [25]. So we need the extra Higgses to be sufficiently massive to suppress $\Delta S = 2$ neutral currents contributions to agree with the experimental data [26, 25, 27]. The actual lower bound on Higgs masses depends on the particular texture chosen for the Yukawa matrices, but can be as low as 120–200 GeV [28].

The paper is organized as follows. In section 2 we will summarize first the structure of Yukawa couplings between twisted fields in the $\mathbb{Z}_3$ orbifold. This will allow us to write the quark and lepton mass matrices. Then, we will see that after the FI breaking, these matrices are substantially modified. In section 3 we analyze the quark masses and mixings and find that without the FI mixing, the CKM matrix cannot be reproduced, but that it can once we take account of the FI mixing. In section 4 we go on to consider the charged lepton and neutrino mass ratios and mixings. Section 5 deals with further constraints coming from the absolute values of the masses, and section 6 summarizes and collates the results.

2. Yukawa couplings and mass matrices in the $\mathbb{Z}_3$ orbifold

As mentioned in the introduction, we are interested in the case that the observable matter lies entirely in the twisted sector, in order to get a realistic mass hierarchy. Thus we first summarize the characteristics of Yukawa couplings between twisted fields in the $\mathbb{Z}_3$ orbifold.

The $\mathbb{Z}_3$ orbifold is constructed by dividing R^6 by the $[\mathrm{SU}(3)]^3$ root lattice modded by the point group (P) with generator θ, where the action of θ on the lattice basis is $\theta e_i = e_{i+1}$, $\theta e_{i+1} = -(e_i + e_{i+1})$ with $i = 1, 3, 5$. The two-dimensional sublattices associated to $[\mathrm{SU}(3)]^3$ are shown in Fig. 1. Let us call $R_k = |e_k|$ and $\alpha_{kl} = \cos\theta_{kl}$, where $e_k e_l = R_k R_l \cos\theta_{kl}$ and $k, l = 1, \ldots, 6$. The initial six-torus of the $\mathbb{Z}_3$ orbifold has 21 degrees of freedom, however taking into account the relations that P-invariance imposes reduces these to only 9 as follows. Indeed, preserving the magnitude of the lattice basis constrains $R_i = R_{i+1}$ and also $\alpha_{i,i+1} = -1/2$ or equivalently $\theta_{i,i+1} = 2\pi/3$. Preserving the angles under the action of P then enforces the two relations $\alpha_{i,j+1} + \alpha_{i+1,j} + \alpha_{i,j} = 0$ and $\alpha_{i,j} = \alpha_{i+1,j+1}$. Only the following nine deformation parameters are left [11]:

$$R_1\,, \qquad R_3\,, \qquad R_5\,, \qquad \alpha_{13}\,, \qquad \alpha_{15}\,, \qquad \alpha_{35}\,, \qquad \alpha_{14}\,, \qquad \alpha_{16}\,, \qquad \alpha_{36}\,. \tag{2.1}$$

In the $\mathbb{Z}_3$ orbifold without deformations the angles between complex planes are vanishing, however this need not be the case. The nine deformation parameters correspond to the VEVs of nine singlet fields that appear in the spectrum of the untwisted sector (of the form $\psi^l_{-1/2}|0\rangle_L \times \tilde{\alpha}^k_{-1}|0\rangle_R$) which have perturbatively flat potentials. These so-called moduli fields are usually denoted by T.

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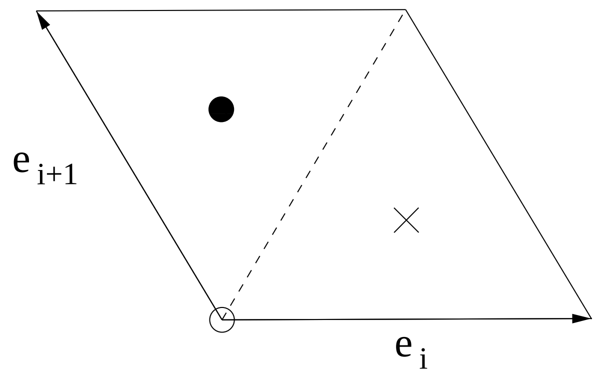

Figure 1: Two dimensional sublattices ($i = 1, 3, 5$) of the $\mathbb{Z}_3$ orbifold. The fixed point components are also shown.

In orbifold constructions, twisted strings appear attached to fixed points under the point group. In the case of the $\mathbb{Z}_3$ orbifold there are 27 fixed points under P, and therefore there are 27 twisted sectors. We will denote the three fixed points of each two-dimensional sublattice as shown in figure 1. For example, the (o x x) fixed point is in the position $f_{oxx} = (1/3)(2e_3 + e_4 + 2e_5 + e_6)$. It was shown in ref. [7] that given two fields associated to two fixed points f_1, f_2, they can only couple to a unique third fixed point f_3 as a consequence of the so-called space group selection rules (thus there are $27\times\ 27\ =\ 729$ allowed Yukawa couplings). In particular, the components of the three fixed points in each sublattice must be either equal or different. For example, if in one of the three sublattices the components of the fixed points f_1 and f_2 are x, x, respectively, the component of the third fixed point f_3 must also be x. If the components for the first two fixed points are x, $\cdot$, then the component for the third fixed point must be o.

The expressions for the different Yukawa couplings can be found for example in [16, appendix]. They contain suppressions factors that depend on the relative positions of the fixed points to which the fields involved in the coupling are attached (i.e. f_1, f_2, f_3) and on the size and shape of the orbifold (i.e. the deformation parameters in eq. (2.1)). In fact, it is possible to show that only 14 couplings out of the 729 allowed are different. In the particular of an orthogonal lattice, i.e. when the six angles in eq. (2.1) are zero, there are only 8 distinct couplings. We will show in the following sections that these three radii are sufficient to fit the whole quark and lepton masses, and so we henceforth restrict the discussion to this case.

We begin by presenting the general form of Yukawa coupling. They are given by the Jacobi theta function,

$$Y_{\theta\theta\theta} = g\ N \sum_{\mathbf{u}\in Z^6} \exp\ [-2\pi(\mathbf{f}_{23} + \mathbf{u})^T M(\mathbf{f}_{23} + \mathbf{u})] \tag{2.2}$$

with

$$N = \sqrt{V}\ \frac{3^{3/4}}{8\pi^3}\frac{\Gamma^6(2/3)}{\Gamma^3(1/3)} \tag{2.3}$$

and

$$M = \begin{pmatrix} T_1 & -(1/2)T_1 & 0 & 0 & 0 & 0 \\ -(1/2)T_1 & T_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_3 & -(1/2)T_3 & 0 & 0 \\ 0 & 0 & -(1/2)T_3 & T_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_5 & -(1/2)T_5 \\ 0 & 0 & 0 & 0 & -(1/2)T_5 & T_5 \end{pmatrix} . \qquad (2.4)$$

Here g is the gauge coupling constant, V is the volume of the unit cell for the $\mathbb{Z}_3$ lattice, and T_i are the diagonal moduli whose real parts are associated to the internal metric $g_{ii} = e_i e_i$, $Re\ T_i = (\sqrt{3}/16\pi^2)R_i^2$, and whose imaginary parts are associated with the torsion. The fact that off-diagonal elements in the matrix M are vanishing is due to our assumption of an orthogonal lattice. The vector $\mathbf{f}_{23}$ represents the six components of $(f_2 - f_3)$. The only eight inequivalent possibilities are

$$(0,0,0,0,0,0)\,, \qquad (2.5)$$

$$\left(\frac{1}{3},\frac{2}{3},0,0,0,0\right), \qquad \left(0,0,\frac{1}{3},\frac{2}{3},0,0\right), \qquad \left(0,0,0,0,\frac{1}{3},\frac{2}{3}\right), \qquad (2.6)$$

$$\left(\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},0,0\right), \qquad \left(\frac{1}{3},\frac{2}{3},0,0,\frac{1}{3},\frac{2}{3}\right), \qquad \left(0,0,\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\right), \qquad (2.7)$$

$$\left(\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\right), \qquad (2.8)$$

corresponding to three fields with the same fixed point components in three sublattices, two sublattices, one sublattice, and no sublattice, respectively.

It turns out that for reasonable values of the moduli, $T_i = \mathcal{O}(1)$, all terms in the sum of eq. (2.2) are negligible with respect to those corresponding to the shortest distance between fixed points. For example, for the third vector in eq. (2.6) the sum, say ε_5, is given by

$$\varepsilon_5 = 3\ e^{-(2\pi/3)T_5}\ (1 + 6\ e^{-2\pi T_1} + 6\ e^{-2\pi T_3} + \cdots)\,, \qquad (2.9)$$

where the dots denote terms with larger suppression factors, which can therefore be approximated as $\epsilon_5 \approx 3\ e^{-(2\pi/3)T_5}$. In the next section we will obtain $T_5 \sim 1.95$ and therefore $\epsilon_5 \sim 0.05$. Taking this into account, the sums in eq. (2.2) corresponding to eqs. (2.5-2.8) can be approximated respectively as

$$1\,, \qquad (2.10)$$

$$\varepsilon_1\,, \qquad \varepsilon_3\,, \qquad \varepsilon_5\,, \qquad (2.11)$$

$$\varepsilon_{13}\,, \qquad \varepsilon_{15}\,, \qquad \varepsilon_{35}\,, \qquad (2.12)$$

$$\varepsilon_{135}\,, \qquad (2.13)$$

where $\varepsilon_i = 3\ e^{-(2\pi/3)T_i}$, $\varepsilon_{ij} = \varepsilon_i\varepsilon_j$, and $\varepsilon_{135} = \varepsilon_1\varepsilon_3\varepsilon_5$. Finally we remark that in the case without deformations, i.e. assuming an orthogonal lattice with $T_i = T$, there are only four

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different couplings, with the sums in eq. (2.2) given by 1, ε, ε^2, ε^3, where $\varepsilon = 3\ e^{-(2\pi/3)T}$. Although we mentioned above that three radii are sufficient to fit the quark and lepton masses, in actual fact this is possible with only one degenerate radius and no deformations at all, as we shall see in the following section. This is of course the most predictive assumption. Conversely, if we begin by allowing three different radii, once we fit the quark and lepton masses, we determine them to be very close.

2.1 Mass matrices before the Fayet-Iliopoulos breaking

With these results we can now turn to the analysis of mass matrices in constructions with two Wilson lines. In what follows we shall consider a particular assignment of Standard Model particles to different fixed points and examine the predictions from the subsequent Yukawa couplings.[3] This is a "bottom up" approach in that we do not explicitly construct the models but are asking which assignment is appropriate for the observed masses and mixings.

Let us first study the situation before taking into account the effect of the FI breaking. Let us suppose that the two non-vanishing Wilson lines (a_1, a_3) correspond to the first and second sublattices. Then the 27 twisted sectors come in nine sets with three equivalent sectors in each one. The three generations of matter (including Higgses) correspond to changing the third sublattice component (x, $\cdot$, o) of the fixed point whilst keeping the other two fixed. Consider for example the following assignments of observable matter to fixed point components in the first two sublattices;

$$\begin{array}{cccccc} Q & o\ o & u^c & o\ o & d^c & x\ o \\ L & \cdot\ \cdot & e^c & \cdot\ x & \nu^c & x\ x \\ & H^u & o\ o & H^d & \cdot\ o & \end{array} \tag{2.14}$$

In this case the up- and down-quark mass matrices, assuming three different radii, are given by

$$M^u = gNA^u\,, \qquad M^d = gN\varepsilon_1 A^d\,, \tag{2.15}$$

where

$$A^u = \begin{pmatrix} v_1^u & v_3^u\varepsilon_5 & v_2^u\varepsilon_5 \\ v_3^u\varepsilon_5 & v_2^u & v_1^u\varepsilon_5 \\ v_2^u\varepsilon_5 & v_1^u\varepsilon_5 & v_3^u \end{pmatrix}, \qquad A^d = \begin{pmatrix} v_1^d & v_3^d\varepsilon_5 & v_2^d\varepsilon_5 \\ v_3^d\varepsilon_5 & v_2^d & v_1^d\varepsilon_5 \\ v_2^d\varepsilon_5 & v_1^d\varepsilon_5 & v_3^d \end{pmatrix}. \tag{2.16}$$

Here v_i^u, v_i^d denote the VEVs of the Higgses H_i^u, H_i^d respectively. For simplicity we will assume for the moment that these VEVs, as well as those of the moduli T_i, are real. Of course, in general they can be complex numbers, and later on we will address the importance that this may have for CP violation.

The elements in the above matrices can be obtained straightforwardly. For example, if the Higgs H_1^u corresponds to (o,o,o), then since the three generations of (3,2) quarks

[3] Note that generally the Standard Model particles *do* belong to different fixed points with such issues as anomaly cancellation being fixed automatically by the string construction.

Q correspond to (o,o,(o,x,·)) and the three generations of ($\bar{3}$,1) quarks u^c to (o,o,(o,x,·)), there are only three allowed couplings,

$$
\begin{aligned}
&(\mathrm{o,o,o})(\mathrm{o,o,o})(\mathrm{o,o,o})\,,\\
&(\mathrm{o,o,o})(\mathrm{o,o,x})(\mathrm{o,o,\cdot})\,,\\
&(\mathrm{o,o,o})(\mathrm{o,o,\cdot})(\mathrm{o,o,x})\,.
\end{aligned}
$$

The corresponding suppression factors are given by 1, ε_5, ε_5 respectively, and are associated with the elements 11, 23, 32 in the matrix M^u. Using the same argument, we obtain the neutrino mass matrix M^ν, as well as the charged-lepton mass matrix M^e,

$$M^\nu = \varepsilon_1\varepsilon_3\; M^u\,, \qquad M^e = \frac{\varepsilon_3}{\varepsilon_1}\; M^d\,. \tag{2.17}$$

2.2 Mass matrices after the Fayet-Iliopoulos breaking

Let us now study how the previous results are modified when one takes into account the FI breaking. As mentioned in the introduction, some scalars, in particular SU(3) × SU(2) singlets, develop large VEVs, of the order of 10^{16-17} GeV, in order to cancel the FI D-term generated by the anomalous U(1). The VEVs of these fields, which we shall denote C_i, break the original gauge group $\mathrm{SU(3)}\times\mathrm{SU(2)}\times\mathrm{U(1)}^n$ down to $\mathrm{SU(3)}_c\times\mathrm{SU(2)}_L\times\mathrm{U(1)}_Y$.

After the breaking, many particles, which we will refer to as ξ, are expected to acquire a high mass because of the generation of effective mass terms. These come for example from operators of the type $C\xi\xi$. In this way vector-like triplets and doublets and also singlets become heavy and disappear from the low-energy spectrum. This is the type of extra matter that typically appears in orbifold constructions. The remarkable point is that the Standard Model matter remain massless, surviving through certain combinations with other states [20]–[23]. Let us consider the simplest example, a model with the Yukawa couplings

$$C_1\xi_1 f\,, \qquad C_2\xi_1\xi_2\,, \tag{2.18}$$

where f denotes a Standard Model field, $\xi_{1,2}$ denote two extra matter fields (triplets, doublets or singlets), and $C_{1,2}$ are the fields developing large VEVs denoted by $\langle C_{1,2}\rangle = c_{1,2}$. It is worth noting here that f can be an u^c, d^c, L, ν^c or e^c field, but not a Q field. This is because in these orbifold models no extra (3,2) representations are present, and therefore the Standard Model field Q cannot mix with other representations through Yukawas.

Clearly the "old" physical particle f will combine with $\xi_{1,2}$. It is now straightforward to diagonalize the mass matrix arising from the mass terms in eq. (2.18) to find two very massive and one massless combination. The latter is given by

$$f' \equiv \frac{1}{\sqrt{|c_1|^2+|c_2|^2}}\,(c_2^* f - c_1^*\xi_2)\,. \tag{2.19}$$

Notice for example that the mass terms (2.18) can be rewritten as $\sqrt{|c_1|^2+|c_2|^2}\;\xi_1\xi_2'$, where $\xi_2' \equiv (1/\sqrt{|c_1|^2+|c_2|^2})\,(c_1 f + c_2\xi_2)$. Indeed the unitary combination is the massless field in eq. (2.19). The Yukawa couplings and hence mass matrices of the effective low energy theory are modified accordingly. For example, consider a model where we begin with a Yukawa coupling HQf. Since we have

$$f = \frac{1}{\sqrt{|c_1|^2 + |c_2|^2}} \left(c_2 f' + c_1^* \xi_2' \right) , \tag{2.20}$$

then the 'new' coupling (involving the light state) will be[4]

$$\frac{c_2}{\sqrt{|c_1|^2 + |c_2|^2}} HQf' \,.$$

The situation in realistic models is more involved since the fields appear in three copies. Thus the mass matrix for the example in eq. (2.18) is (using the results above)

$$gN \,(\xi_1^1\ \xi_1^2\ \xi_1^3) \left\{ \varepsilon' \begin{pmatrix} c_1^1 & c_1^3\varepsilon_5 & c_1^2\varepsilon_5 \\ c_1^3\varepsilon_5 & c_1^2 & c_1^1\varepsilon_5 \\ c_1^2\varepsilon_5 & c_1^1\varepsilon_5 & c_1^3 \end{pmatrix} \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix} + \varepsilon'' \begin{pmatrix} c_2^1 & c_2^3\varepsilon_5 & c_2^2\varepsilon_5 \\ c_2^3\varepsilon_5 & c_2^2 & c_2^1\varepsilon_5 \\ c_2^2\varepsilon_5 & c_2^1\varepsilon_5 & c_2^3 \end{pmatrix} \begin{pmatrix} \xi_2^1 \\ \xi_2^2 \\ \xi_2^3 \end{pmatrix} \right\} , \tag{2.21}$$

where $\varepsilon', \varepsilon''$ can take different values

$$\varepsilon', \varepsilon'' \equiv 1\,,\ \varepsilon_1\,,\ \varepsilon_3\,,\ \varepsilon_1\varepsilon_3\,, \tag{2.22}$$

depending on the particular case. For example, if the field f corresponds to the down quark with the assignment as in eq. (2.14), and the fields C_1, C_2, ξ_1, ξ_2 have the assignments for the first two sublattices (x o), (o x), (x o), $(\cdot\ \cdot)$, respectively, then $\varepsilon' = 1$ and $\varepsilon'' = \varepsilon_1\varepsilon_3$.

Now, in order to simplify the analysis, let us consider the following VEVs for the $C_{1,2}^i$ fields;[5]

$$\begin{aligned} c_1^1 &\equiv c_1\,, & c_1^2 &= c_1^3 = 0\,, \\ c_2^1 &= c_2^3 = 0\,, & c_2^2 &\equiv c_2\,. \end{aligned} \tag{2.23}$$

Then eq. (2.21) gives rise to the mass terms

$$\sqrt{|\hat{c}_1|^2 + |\hat{c}_2\varepsilon_5|^2}\ \xi_1^1\xi_2'^1 \ + \ \sqrt{|\hat{c}_1\varepsilon_5|^2 + |\hat{c}_2|^2}\ \xi_1^2\xi_2'^2 \ + \ \varepsilon_5\sqrt{|\hat{c}_1|^2 + |\hat{c}_2|^2}\ \xi_1^3\xi_2'^3\,, \tag{2.24}$$

where

$$\begin{aligned} \xi_2'^1 &\equiv \frac{1}{\sqrt{|\hat{c}_1|^2 + |\hat{c}_2\varepsilon_5|^2}} \left(\hat{c}_1 f_1 + \hat{c}_2\varepsilon_5\xi_2^3 \right) , \\ \xi_2'^2 &\equiv \frac{1}{\sqrt{|\hat{c}_1\varepsilon_5|^2 + |\hat{c}_2|^2}} \left(\hat{c}_1\varepsilon_5 f_3 + \hat{c}_2\xi_2^2 \right) , \\ \xi_2'^3 &\equiv \frac{1}{\sqrt{|\hat{c}_1|^2 + |\hat{c}_2|^2}} \left(\hat{c}_1 f_2 + \hat{c}_2\xi_2^1 \right) , \end{aligned} \tag{2.25}$$

[4]We should add that the coupling $HQ\xi_2$, which would induce another contribution to HQf', is not in fact allowed. For this to be the case the fields ξ_2 and f would have had to have exactly the same $\mathrm{U}(1)^n$ charges. This is not possible since different particles all have different gauge quantum numbers.

[5]In principle we are allowed to do this since the cancellation of the FI D-term only imposes $\sum_i Q_i^{(a)}(|c_i^1|^2 + |c_i^2|^2 + |c_i^3|^2) = const$, where $Q_i^{(a)}$ are the charges of the C_i fields under the anomalous U(1), and therefore flat directions arise. As for the T-moduli, these VEVs can eventually be determined dynamically through supersymmetry breaking. For attempts in this direction see e.g. ref. [29].

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and

$$\hat{c}_1 \equiv \varepsilon' c_1 \,, \qquad \hat{c}_2 \equiv \varepsilon'' c_2 \,. \tag{2.26}$$

Following the discussion for eqs. (2.19) and (2.20) we can deduce straightforwardly that the new mass matrices for the quarks are

$$\mathcal{M}^u = a^{u^c} M^u B^{u^c} \,, \qquad \mathcal{M}^d = a^{d^c} M^d B^{d^c} \,, \tag{2.27}$$

where[6]

$$a^f = \frac{\hat{c}_2^f}{\sqrt{|\hat{c}_1^f|^2 + |\hat{c}_2^f|^2}} \,, \tag{2.28}$$

M^u, M^d are given in eq. (2.15), and

$$B^f = \begin{pmatrix} \beta^f \varepsilon_5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^f/\varepsilon_5 \end{pmatrix} , \tag{2.29}$$

with

$$\alpha^f = \varepsilon_5 \sqrt{\frac{|\hat{c}_1^f|^2 + |\hat{c}_2^f|^2}{|\hat{c}_1^f \varepsilon_5|^2 + |\hat{c}_2^f|^2}} \,, \qquad \beta^f = \sqrt{\frac{|\hat{c}_1^f|^2 + |\hat{c}_2^f|^2}{|\hat{c}_1^f|^2 + |\hat{c}_2^f \varepsilon_5|^2}} \,. \tag{2.30}$$

Here we have already taken into account that different fields will couple to different C_i fields, and therefore we will generally have $c_{1,2}^{u^c} \neq c_{1,2}^{d^c}$.

Concerning the leptons, we have two possible structures for the mass matrices. If we assume that only the fields ν^c and e^c mix with other representations through Yukawas, the situation is similar to that above, generating the following mass matrices;

$$\mathcal{M}^\nu = a^{\nu^c} M^\nu B^{\nu^c} \,, \qquad \mathcal{M}^e = a^{e^c} M^e B^{e^c} \,, \tag{2.31}$$

where M^ν, M^e are given in eq. (2.17). However, in principle, the left handed leptons L may also mix with other representations, which instead gives rise to the matrices

$$\mathcal{M}^\nu = a^L a^{\nu^c} B^L M^\nu B^{\nu^c} \,, \qquad \mathcal{M}^e = a^L a^{e^c} B^L M^e B^{e^c} \,. \tag{2.32}$$

We will find that the second possibility is the one which is consistent with the observed masses.

For a given field there are basically three patterns for the values of α and β. When $\hat{c}_1 \sim \hat{c}_2$ one obtains $\alpha \sim \varepsilon_5$ and $\beta \sim 1$, since in both denominators in eq. (2.30) the term with ε_5 is negligible. In addition $a \sim 1$. The possibility with $\hat{c}_1 \ll \hat{c}_2$ may also be present and turns out to give a trivial result. For example, this is the case when $\varepsilon'' = 1$, $\varepsilon' = \varepsilon_1 \varepsilon_3$, and therefore using eq. (2.26) one obtains $\hat{c}_2 = c_2$ and $\hat{c}_1 = c_1 \varepsilon_1 \varepsilon_3$. Since one

[6] Note that, although $c_{1,2}$ are in general complex VEVs, they only introduce a global and therefore unphysical phase in the mass matrix. Of course, this is an artifact of the direction (2.23) choosen to cancel the FI D-term. More complicated directions would give rise in principle to a contribution to the CP phase. This mechanism to generate the CP phase through the VEVs of the fields cancelling the FI D-term was used first, in the context of non-renormalizable couplings, in ref. [11]. For a recent analysis see ref. [30].

expects $c_1 \sim c_2$, as obtained in explicit models [20]–[23], $\hat{c}_1$ is much smaller than $\hat{c}_2$. As a consequence, $\alpha \approx \varepsilon_5$, $\beta \approx 1/\varepsilon_5$, and therefore $B \sim 1$. Finally, the third pattern arises when $\hat{c}_1 \gg \hat{c}_2$, i.e. $\varepsilon' \gg \varepsilon''$. In this case one gets

$$\alpha \sim 1\,, \qquad \beta \sim 1\,. \tag{2.33}$$

In addition,

$$a \sim \frac{c_2}{c_1}\frac{\varepsilon''}{\varepsilon'} \sim \frac{\varepsilon''}{\varepsilon'}\,, \tag{2.34}$$

where taking into account eq. (2.22), the above ratio can take the values $\varepsilon''/\varepsilon' = \varepsilon_1, \varepsilon_3, \varepsilon_1\varepsilon_3$. An example of the above situation is given by $\varepsilon' = 1$ and $\varepsilon'' = \varepsilon_1\varepsilon_3$. There is a subtlety in some cases, as for example when $\varepsilon' = 1$ and $\varepsilon'' = \varepsilon_1$, since then

$$\alpha \sim \varepsilon_5 \sqrt{\frac{|c_1|^2}{|c_1\varepsilon_5|^2 + |c_2\varepsilon_1|^2}} \sim \frac{\varepsilon_5}{\sqrt{|\varepsilon_5|^2 + |\varepsilon_1|^2}}\,. \tag{2.35}$$

Thus $\alpha \sim \varepsilon_5/\varepsilon_1$ if $\varepsilon_1 \gg \varepsilon_5$, and therefore the element 33 in B above would be $1/\varepsilon_1$ instead of $1/\varepsilon_5$. On the other hand, one does not expect an 'asymmetric' supersymmetry breaking to occur naturally, and therefore the moduli T_i and hence the ε_i should be of the same order. As we have already mentioned, we will determine the ε_i in the next section using the available information from quark masses and mixing angles, eqs. (1.1) and (1.2), and will indeed find that the most attractive solution is where all of them are of the same order. Finally, it is worth noting that the pattern giving rise to values (2.33) will be the most successful one for the quark masses, as we will discuss in the next section. For the leptons the first pattern will also be interesting, as we will see in section 4.

Let us add that permutations in the diagonal elements of the above matrix (2.29) are also possible. This is because as well as the direction in eq. (2.23) we have five additional non-trivial simple possibilities. For example, we could instead have assumed $c_1^1 = c_1$, $c_2^3 = c_2$, with all other VEVs vanishing, and then the elements 22 and 33 in (2.29) would have been permuted. We will see below that this additional degree of freedom is helpful in reproducing the observed neutrino masses.

3. Quark mass ratios and mixings

One nice feature of the models we are interested in is that the Yukawas have a similar form modulo overall factors. This means that, for example, the mass ratios of the first/second generation downs are related to those of the first/second generation leptons, and there is considerable predictive power. Therefore it makes sense to determine mass ratios and mixings before dealing with the various overall prefactors.

We will try first to extract information from eq. (2.15). We will show that although the observed quark mass ratios and Cabibbo angle can be reproduced correctly, the 13 and 23 elements of the CKM matrix in eq. (1.2) cannot be obtained. Fortunately, this is not the end of the story. We will see that the matrices obtained after FI breaking, modified with the contribution in eq. (2.29) will improve the result, giving rise to the correct elements for the CKM matrix.

3.1 Before the Fayet-Iliopoulos breaking

First consider the quark mass matrices (2.15) before taking into account the impact of the FI breaking. Clearly these matrices are very constrained, and indeed it is possible to see immediately that they are incompatible with a successful CKM matrix.

In order to prove this we can use the following procedure to find the CKM matrix. The symmetric matrix A of eq. (2.16) can be diagonalized by a matrix V

$$A_{\rm diag} = V A\, V^T\,, \tag{3.1}$$

where we can define V as orthogonal rotations,

$$V = R_{12}\; R_{23}\; R_{13}\,, \tag{3.2}$$

through angles ϕ_{12}, ϕ_{23}, ϕ_{13} . Now we can write the rotations as an expansion in ε_5,

$$\sin\phi_{ij} = a_{ij}\varepsilon_5 + b_{ij}\varepsilon_5^2 + c_{ij}\varepsilon_5^3 + \cdots\,, \tag{3.3}$$

and solve for $(VAV^T)_{ab} = 0$, where $a \neq b$, order by order in ε_5 to the desired accuracy, deducing the coefficients a_{ij}, b_{ij}, c_{ij} as we go along. The CKM matrix is then

$$V_{CKM} = V_U\; V_D^T\,. \tag{3.4}$$

Assuming ε is sufficiently small, which we check presently, to first order in epsilon the eigenvalues and diagonalization of A goes as

$$V A\, V^T = \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix} + \mathcal{O}(\varepsilon_5)\,, \tag{3.5}$$

where

$$V = \begin{pmatrix} 1 & \frac{v_3}{v_1-v_2}\varepsilon_5 & \frac{v_2}{v_1-v_3}\varepsilon_5 \\ -\frac{v_3}{v_1-v_2}\varepsilon_5 & 1 & \frac{v_1}{v_2-v_3}\varepsilon_5 \\ -\frac{v_2}{v_1-v_3}\varepsilon_5 & -\frac{v_1}{v_2-v_3}\varepsilon_5 & 1 \end{pmatrix} + \mathcal{O}(\varepsilon_5^2)\,. \tag{3.6}$$

Without FI breaking therefore, the mass hierarchies must be provided entirely by the Higgs VEVs, and we have

$$\begin{aligned} \{v_1^u,\, v_2^u,\, v_3^u\} &\propto \{m_u,\, m_c,\, m_t\}\,, \\ \{v_1^d,\, v_2^d,\, v_3^d\} &\propto \{m_d,\, m_s,\, m_b\}\,. \end{aligned} \tag{3.7}$$

The CKM matrix is then given by

$$V_{CKM} \approx \begin{pmatrix} 1 & \varepsilon_5\,(m_b/m_s - m_t/m_c) & \varepsilon_5\,(m_s/m_b - m_c/m_t) \\ -\varepsilon_5\,(m_b/m_s - m_t/m_c) & 1 & \varepsilon_5\,(m_d/m_b - m_u/m_t) \\ -\varepsilon_5\,(m_s/m_b - m_c/m_t) & -\varepsilon_5\,(m_d/m_b - m_u/m_t) & 1 \end{pmatrix} + O(\varepsilon_5^2)\,. \tag{3.8}$$

From $(V_{CKM})_{12} \approx 0.22$, and taking for example $m_u = 4.5\,\mathrm{MeV}$, $m_d = 8.5\,\mathrm{MeV}$, $m_s = 100\,\mathrm{MeV}$, $m_c = 1.35\,\mathrm{GeV}$, $m_b = 4.5\,\mathrm{GeV}$ and $m_t = 175\,\mathrm{GeV}$ we determine $\varepsilon_5 \approx 2.6\times 10^{-3}$ (warranting our assumption of small ε) and hence

$$(V_{CKM})_{13} \approx 3.8\times 10^{-5}\,, \qquad (V_{CKM})_{23} \approx 4.8\times 10^{-6}\,. \tag{3.9}$$

Thus, quite generally, the model without FI breaking fails already at the quark mixing stage.

3.2 After the Fayet-Iliopoulos breaking

As discussed in subsection 2.2, in all realistic models constructed the standard-model matter survives through certain combinations with other states [20]–[23]. Taking this into account we will see that the above results can be modified. In particular, it will be possible to get the right spectrum and a CKM matrix with the right form. For the quarks the final mass matrices are now given as in eq. (2.27)

$$\mathcal{M}^u = gNa^{u^c}A^uB^{u^c}\,, \qquad \mathcal{M}^d = gN\varepsilon_1 a^{d^c}A^dB^{d^c}\,, \tag{3.10}$$

where

$$A\;B = \begin{pmatrix} v_1\varepsilon_5\beta & v_3\varepsilon_5 & v_2\alpha \\ v_3\varepsilon_5^2\beta & v_2 & v_1\alpha \\ v_2\varepsilon_5^2\beta & v_1\varepsilon_5 & v_3\alpha/\varepsilon_5 \end{pmatrix}. \tag{3.11}$$

As discussed below eq. (2.29), there are three possible patterns for B depending on the particular Yukawa couplings producing the combination of the fields. The matrix B with values as in eq. (2.33) will be the interesting one. The reason this choice works well is that the hierarchy in the masses is now driven by the B matrix which aligns the Yukawas into hierarchical columns. To leading order in ε_5 the eigenvalues of the matrix (3.11) are given by

$$\left\{ v_1\varepsilon_5\beta - \frac{(v_3\alpha)^2\varepsilon_5^3}{v_2}\;,\; v_2\;,\; \frac{v_3\alpha}{\varepsilon_5} \right\}. \tag{3.12}$$

As we will need a slight hierarchy in the VEVs, $v_3 > v_2 > v_1$, we have kept the ε_5^3 term in the lightest eigenvalue. We will see that this will be negligible for the down quark but not for the up quark. With the approximate eigenvalues above we then have

$$\{v_1^u\;, v_2^u\;, v_3^u\} \propto \left\{ \frac{1}{\varepsilon_5\beta^{u^c}}\left(m_u + \varepsilon_5^2\,\frac{m_t^2}{m_c}\right), m_c, \frac{m_t\varepsilon_5}{\alpha^{u^c}} \right\}, \tag{3.13}$$

$$\left\{v_1^d\;, v_2^d\;, v_3^d\right\} \propto \left\{ \frac{1}{\varepsilon_5\beta^{d^c}}\left(m_d + \varepsilon_5^2\,\frac{m_b^2}{m_s}\right), m_s\;, \frac{m_b\varepsilon_5}{\alpha^{d^c}} \right\}. \tag{3.14}$$

In order to find the CKM matrix we can use the same procedure as in the previous subsection. The full expression for the CKM matrix is independent of $\alpha's$ and $\beta's$, and in fact the role of the B matrices is simply to modify the eigenvalues. To leading order in ε_5 the CKM matrix becomes

$$V_{CKM} = \begin{pmatrix} 1 & \left(v_3^d/v_2^d - v_3^u/v_2^u\right)\varepsilon_5 & \left(v_2^d/v_3^d - v_2^u/v_3^u\right)\varepsilon_5 \\ -\left(v_3^d/v_2^d - v_3^u/v_2^u\right)\varepsilon_5 & 1 & \left(v_1^d/v_3^d - v_1^u/v_3^u\right)\varepsilon_5 \\ -\left(v_2^d/v_3^d - v_2^u/v_3^u\right)\varepsilon_5 & -\left(v_1^d/v_3^d - v_1^u/v_3^u\right)\varepsilon_5 & 1 \end{pmatrix} + \mathcal{O}(\varepsilon_5^2). \tag{3.15}$$

For the calculation of the 13 element of the CKM matrix, $(V_{CKM})_{13}$, we require an accuracy of ε_5^2. The additional piece is given by

$$(V_{CKM})_{13} = \left(\frac{v_2^d}{v_3^d} - \frac{v_2^u}{v_3^u}\right)\varepsilon_5 + \left(\frac{v_1^u}{v_2^u} - \frac{v_1^d}{v_3^d}\frac{v_3^u}{v_2^u}\right)\varepsilon_5^2 + \cdots\,. \tag{3.16}$$

Using eqs. (3.13) and (3.14) we obtain

$$
\begin{aligned}
(V_{CKM})_{12} &= -\varepsilon_5^2 \left(\frac{m_t}{m_c} \frac{1}{\alpha^{u^c}} - \frac{m_b}{m_s} \frac{1}{\alpha^{d^c}} \right) , \\
(V_{CKM})_{23} &= \frac{1}{\varepsilon_5} \left(\frac{1}{m_b} \left[m_d + \varepsilon_5^5 \frac{m_b^2}{m_s} \right] \frac{\alpha^{d^c}}{\beta^{d^c}} - \frac{1}{m_t} \left[m_u + \varepsilon_5^5 \frac{m_t^2}{m_c} \right] \frac{\alpha^{u^c}}{\beta^{u^c}} \right) , \\
(V_{CKM})_{13} &= \left(\frac{m_s}{m_b} \alpha^{d^c} - \frac{m_c}{m_t} \alpha^{u^c} \right) - \\
&\quad - \varepsilon_5 \left(\frac{1}{m_b} \left[m_d + \varepsilon_5^5 \frac{m_b^2}{m_s} \right] \frac{m_t}{m_c} \frac{\alpha^{d^c}}{\alpha^{u^c} \beta^{d^c}} - \frac{1}{m_c} \left[m_u + \varepsilon_5^5 \frac{m_t^2}{m_c} \right] \frac{1}{\beta^{u^c}} \right) . \qquad (3.17)
\end{aligned}
$$

The expression for $(V_{CKM})_{13}$ is written to order ε_5^2 while the other two elements are sufficiently accurate to leading order in ε_5 (note that the mass eigenvalues, written above, carry powers of ε_5). We therefore have two predictions for the CKM elements and can eliminate ε_5 using the equation for $(V_{CKM})_{12}$;

$$
\varepsilon_5 = \sqrt{\frac{(V_{CKM})_{12}}{((m_t/m_c)\ 1/\alpha^{u^c} - (m_b/m_s)\ 1/\alpha^{d^c})}} . \qquad (3.18)
$$

For instance, taking $\alpha^{u^c,d^c} \sim 1$ as discussed in eq. (2.33), and the same numerical example as the one below eq. (3.8) we find that

$$
\varepsilon_5 \approx 0.05 . \qquad (3.19)
$$

Since $\varepsilon_5 = 3\ e^{-(2\pi/3)T_5}$, this value corresponds to $T_5 \approx 1.95$. With this result we can check now that indeed, as mentioned above, the term proportional to ε_5^2 is negligible in eq. (3.14) but not in eq. (3.13).

Using the above value for ε_5, and $\beta^{u^c,d^c} \sim 1$, we can also compute now the elements 23 and 13 of the CKM matrix in eq. (3.17) with the result:

$$
(V_{CKM})_{23} \approx 0.038 , \qquad (V_{CKM})_{13} \approx 0.0026 . \qquad (3.20)
$$

It is worth noting that the first piece of $(V_{CKM})_{13}$ in eq. (3.17) has the value 0.014, a factor of three too large, but it cancels against the the second piece resulting in the correct value. The values of eq. (3.20) are in quite good agreement with the experimental ones in eq. (1.2), considering also that we are assuming all $\alpha's$ and $\beta's$ equal one, particular values for quark masses, and neglecting any renormalization effects (eq. (2.15) corresponds to Yukawa matrices at the string scale). For the sake of comparison, in addition to the CKM matrix to first order (with a 2^{nd} order 13 element) in ε_5 obtained above,

$$
V_{CKM} = \begin{pmatrix} 1 & 0.22 & 0.003 \\ 0.22 & 1 & 0.038 \\ 0.01 & 0.038 & 1 \end{pmatrix} , \qquad (3.21)
$$

we also show it to second order

$$
V_{CKM} = \begin{pmatrix} 0.976 & 0.222 & 0.003 \\ 0.222 & 0.975 & 0.041 \\ 0.010 & 0.038 & 0.999 \end{pmatrix} , \qquad (3.22)
$$

and the one computed numerically

$$V_{CKM} = \begin{pmatrix} 0.978 & 0.210 & 0.002 \\ 0.210 & 0.978 & 0.039 \\ 0.010 & 0.038 & 0.999 \end{pmatrix}. \tag{3.23}$$

Note that the discrepancy between the numerical and analytic values of $(V_{CKM})_{12}$ is because here we are still using the first order determination of ε_5 in eq. (3.18) in all the expressions.

In addition to the magnitudes of the CKM matrix elements we also require a CP violating phase. In string theory, CP violation is problematic because CP is a gauge symmetry of the full theory. There has been continued interest in how it can be spontaneously broken so that the resulting CP phases are physically observable [11, 31, 30, 32, 33, 34]. Since we are considering entirely renormalizable Yukawa couplings, there appear to be only two possibilities here (in addition to the one already mentioned in footnote 6). First one can assume that the VEVs of the moduli have an imaginary phase, which can occur when the flat moduli directions are lifted by supersymmetry breaking and find their minimum where the phases are non-zero [31]. Such a phase feeds directly into ε_5. It is easy to check that this phase is physically observable, and leads to a non-zero δ phase for the CKM matrix which is of order one. An alternative way to break CP has been explored in type II D-brane models, which is to break CP without breaking supersymmetry by introducing discrete torsion [33] or Wilson lines [34]. (Torsion would require a factor $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold. To our knowledge CP violation from torsion has not yet been examined for the heterotic case.)

Let us conclude this section by using the mass ratios to fix the relative Higgs VEVs. Our results imply that they take the following values;

$$v_3^d \approx v_2^d \approx v_1^d \,, \qquad v_3^u \approx 6v_2^u \approx 36v_1^u \,, \tag{3.24}$$

where we have used eqs. (3.13) and (3.14). Since the electroweak symmetry breaking condition,

$$(v_1^u)^2 + (v_2^u)^2 + (v_3^u)^2 + (v_1^d)^2 + (v_2^d)^2 + (v_3^d)^2 = 2\left(\frac{M_W}{g_2}\right)^2 \,, \tag{3.25}$$

must be fulfilled, using eq. (3.24) we obtain

$$37(v_2^u)^2 + 3(v_2^d)^2 \approx (174\,\mathrm{GeV})^2 \,. \tag{3.26}$$

It is striking that the three v_i^d VEVs have to be degenerate to satisfy the experimental values. This is in contrast to the case without FI breaking, where the Higgs VEVs had to have the hierarchies observed in the fermion masses. Here the masses are provided by the hierarchical mixing of the physical fields with the FI fields. This degeneracy in Higgs VEVs will be advantageous from the model building point of view when it comes to the lepton sector, as it allows us to permute the elements of the B^L B^{e^c} and B^ν without affecting the charged lepton masses. This cuts down the number of possibilities for the charged lepton masses, thereby increasing predictivity, but allows us more freedom to manipulate the neutrino mass matrices, as we now see.

4. Lepton mass ratios and mixings

The mass matrices for leptons before the FI breaking, given in eq. (2.17), are as for the quarks, but with different prefactors. Thus the system is extremely constrained and we will see that the correct masses cannot be obtained if we only include B^{e^c} and B^{ν^c} in the mass matrices. However, unlike the quarks where B^Q cannot be present, here we have the possibility of including a mixing for the left handed fields as well by including B^L in the analysis. We will see that this improves the results giving the correct masses for charged leptons. For neutrinos this will not be sufficient, but a see-saw mechanism with a reduced see-saw scale breaking will solve the problem.

4.1 Charged leptons

Consider first the lepton masses with an FI breaking B matrix for just the right handed fields, B^{e^c}. Using eq. (2.31) we have

$$\mathcal{M}^e = gN\epsilon_3 a^{e^c} A^d B^{e^c} = gN\epsilon_3 a^{e^c} \begin{pmatrix} v_1^d \varepsilon_5 \beta^{e^c} & v_3^d \varepsilon_5 & v_2^d \alpha^{e^c} \\ v_3^d \varepsilon_5^2 \beta^{e^c} & v_2^d & v_1^d \alpha^{e^c} \\ v_2^d \varepsilon_5^2 \beta^{e^c} & v_1^d \varepsilon_5 & v_3^d \alpha^{e^c}/\varepsilon_5 \end{pmatrix}. \tag{4.1}$$

Thus the masses are given by

$$\{m_e, m_\mu, m_\tau\} \propto \left\{ v_1^d \varepsilon_5 \beta^{e^c} \,,\, v_2^d \,,\, \frac{v_3^d \alpha^{e^c}}{\varepsilon_5} \right\}. \tag{4.2}$$

Since the down-quark masses are determined by eq. (3.12), the ratios are already very constrained. Comparing first to second generation masses gives

$$\frac{\beta^{e^c}}{\beta^{d^c}} = \frac{m_s}{m_d} \frac{m_e}{m_\mu} \sim 0.05\,. \tag{4.3}$$

However, pattern (2.33) used above in order to obtain the correct quark mass ratios and mixings implies that $\beta^{d^c} \sim 1$ and therefore that $\beta^{e^c} \sim 0.05 \sim \epsilon_5$. The latter is in contradiction with the three allowed patterns for α's and β's discussed below eq. (2.32). We can try to modify some of the assumptions, by for example permuting the entries of the B^{e^c} with respect to those of B^{d^c}, but in fact this makes the ratios worse, and no modification yields charged-leptons masses in a natural way.

Fortunately, as we have already seen, the natural situation is for both the left handed and right handed leptons to combine with other fields. In this case we should introduce another matrix, B^L, for the left handed leptons. Now using eq. (2.32) we obtain

$$\mathcal{M}^e = gN\epsilon_3 a^L a^{e^c} B^L A^d B^{e^c} = gN\epsilon_3 a^L a^{e^c} \begin{pmatrix} v_1^d \varepsilon_5^2 \beta^L \beta^{e^c} & v_3^d \varepsilon_5 & v_2^d \alpha^L \alpha^{e^c}/\epsilon_5 \\ v_3^d \varepsilon_5^3 \beta^L \beta^{e^c} & v_2^d & v_1^d \alpha^L \alpha^{e^c}/\epsilon_5 \\ v_2^d \varepsilon_5^3 \beta^L \beta^{e^c} & v_1^d \varepsilon_5 & v_3^d \alpha^L \alpha^{e^c}/\varepsilon_5^2 \end{pmatrix}. \tag{4.4}$$

The masses are now given by

$$\{m_e, m_\mu, m_\tau\} \propto \left\{ \beta^L \beta^{e^c} v_1^d \varepsilon_5^2 \,,\, v_2^d \,,\, \frac{\alpha^L \alpha^{e^c} v_3^d}{\varepsilon_5^2} \right\}. \tag{4.5}$$

Concentrating on the electron/down ratios again we now find

$$\frac{\beta^L \beta^{e^c} \varepsilon_5}{\beta^{d^c}} = \frac{m_s}{m_d}\frac{m_e}{m_\mu} \sim 0.05\,, \tag{4.6}$$

which is remarkably close to the correct value if $\beta^{L,e^c} \sim 1$. Comparing second to third generation masses we also find

$$\frac{\alpha^{d^c}\varepsilon_5}{\alpha^L \alpha^{e^c}} = \frac{m_\tau}{m_\mu}\ \frac{m_s}{m_b} \sim 1\,, \tag{4.7}$$

so that we require

$$\alpha^L \alpha^{e^c} \sim \varepsilon_5 \alpha^{d^c}\,. \tag{4.8}$$

If we keep $\alpha^{d^c} \sim 1$ to preserve our good CKM prediction, this is quite a mild requirement on α^L and α^{e^c}, since we just need $\alpha^L \alpha^{e^c} \sim \epsilon_5$. To satisfy this we need only recall that the two non-trivial patterns of B-matrix have $\alpha \sim 1$, $\beta \sim 1$ or $\alpha \sim \varepsilon_5$, $\beta \sim 1$. We therefore require that B^L and B^{e^c} are of the opposite types.

4.2 Neutrinos

4.2.1 Dirac Neutrino masses with no see-saw

Turning now to the neutrinos, using eq. (2.32) one obtains the following mass matrix;

$$\mathcal{M}^\nu = gN\varepsilon_1\varepsilon_3 a^L a^{\nu^c} B^L A^u B^{\nu^c} = gN\varepsilon_1\varepsilon_3 a^L a^{\nu^c} \begin{pmatrix} v_1^u \varepsilon_5^2 \beta^L \beta^{\nu^c} & v_3^u \varepsilon_5 & v_2^u \alpha^L \alpha^{\nu^c}/\varepsilon_5 \\ v_3^u \varepsilon_5^3 \beta^L \beta^{\nu^c} & v_2^u & v_1^u \alpha^L \alpha^{\nu^c}/\varepsilon_5 \\ v_2^u \varepsilon_5^3 \beta^L \beta^{\nu^c} & v_1^u \varepsilon_5 & v_3^u \alpha^L \alpha^{\nu^c}/\varepsilon_5^2 \end{pmatrix}. \tag{4.9}$$

Thus the neutrino masses are given by

$$\left\{m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}\right\} \propto \left\{\beta^L \beta^{\nu^c} v_1^u \varepsilon_5^2\ , v_2^u\ , \frac{\alpha^L \alpha^\nu v_3^u}{\varepsilon_5^2}\right\}, \tag{4.10}$$

and consequently we obtain the ratios

$$\begin{aligned} \frac{m_c}{m_u}\frac{m_{\nu_e}}{m_{\nu_\mu}} &= \frac{\beta^L \beta^{\nu^c} \varepsilon_5}{\beta^{u^c}} \sim \varepsilon_5\,, \\ \frac{m_c}{m_t}\frac{m_{\nu_\tau}}{m_{\nu_\mu}} &= \frac{\alpha^L \alpha^{\nu^c}}{\alpha^{u^c}\varepsilon_5} \sim \frac{\alpha^{\nu^c}}{\alpha^{e^c}}\,, \end{aligned} \tag{4.11}$$

where we have used the results for α's and β's derived from the quarks and charged leptons. We shall assume that the experimental data on neutrino mass-squared differences reflects their actual masses; i.e. we assume that neutrino masses are hierarchical.

Now consider the second relation in eq. (4.11). The neutrino hierarchy is of the order of 10, and $m_c/m_t \approx 10^{-2}$ therefore the natural choice would be $\alpha^{\nu^c} \sim \alpha^L \sim \varepsilon_5$, $\alpha^{e^c} \sim 1$ which determines, $a^{\nu^c} \sim 1$ and $a^L \sim 1$. The heaviest neutrino mass is then of order 500 MeV. Even if the neutrino hierarchies had suggested $a^{\nu^c} \sim \alpha^L \sim \varepsilon_5$ the largest neutrino mass would still have been 1 MeV, about the same as the electron mass. Again we should remember that the positions of the $\beta\varepsilon_5$ and α/ε_5 in the mixing matrix B^L were not fixed by the charged leptons since the H^d VEVs are degenerate. In other words, because v_i^d are degenerate we may permute the positions of the v_i^u in the above expressions without changing the charged lepton masses. However the largest hierarchy in the v^u is ≈ 36 which is not large enough to bring m_{ν_τ} to the required values.

4.2.2 Neutrinos masses via the see-saw

We can solve the above problem through a see-saw mechanism [35], but because the Dirac masses are already significantly suppressed, we expect that the required see-saw scale will be lower than the usual one. In order to avoid introducing any ad-hoc scales into the model we would ideally like the see-saw scale to be associated with some other scale already existing in the model. At this point we note a striking coincidence; if the Yukawa coupling for the neutrino is of order m_e and the see-saw scale is 1 TeV, then the expected neutrino mass is

$$\frac{m_e^2}{1\,\mathrm{TeV}} = 0.25\,\mathrm{eV}\,, \tag{4.12}$$

which is within an order of magnitude of the experimental values. This suggests that the most natural situation is one in which a see-saw mass of order a few TeV is generated by the electroweak symmetry breaking. Unfortunately for the examples we examine here, we will find that the see-saw scale is two orders of magnitude too large, but we think this is an intriguing possibility that is worth pursuing. The first guess for the neutrino see-saw superpotential is then

$$W^\nu \sim H^u L \nu^c + S\nu^c\nu^c\,, \tag{4.13}$$

where S is the same singlet that dynamically generates the μ term through the coupling SH^uH^d. (Note in this context that the Giudice–Masiero mechanism to generate a μ term through the Kähler potential is not available for prime orbifolds such as the $\mathbb{Z}_3$ orbifold [36].) Therefore S is expected to get a VEV of order 1 TeV. This is clearly a supersymmetric version of the see-saw mechanism, but with the new feature that the see-saw scale is tied to the weak scale.[7] We can make the hierarchy in the neutrinos less steep by permuting the mixing to the heavy fields, so that we will be able to generate the neutrino masses in the same framework without introducing any hierarchies.

Unfortunately the superpotential is slightly more complicated than that in eq. (4.13) because not all the couplings are allowed. In fact, for the Z_3 model under discussion, the particle assignment in eq. (2.14) which can give a nice suppression to the tau and bottom masses does not allow the $S\nu^c\nu^c$ coupling directly. This is because the presence of the coupling SH^uH^d implies that S must be assigned to the following fixed point components in the first two sublattices:

$$S \quad x\ o\,. \tag{4.14}$$

[7] It is worth pointing out here that the same mechanism could be used in the context of the non-supersymmetric Standard Model, using simply a Majorana mass for the right-handed neutrino of order 1 TeV.

On the other hand the coupling $S\nu^c\nu^c$ can only be allowed if S is assigned to the same fixed point components as ν^c in the first two sublattices;

$$S \quad x\; x\, . \tag{4.15}$$

However, it is always possible to couple indirectly through other singlets living at different fixed points. In that case we can still generate heavy Majorana masses for the neutrinos, by introducing two more singlets S' and S'' and modifying the superpotential to

$$SH^uH^d + SS'S'' + S'\nu^c\nu^c\, . \tag{4.16}$$

This case, where we are using the assignment

$$S \quad x\; o \qquad\quad S' \quad x\; x \qquad\quad S'' \quad x\; \cdot \tag{4.17}$$

is an obvious generalization of our first guess.[8] Under this assumption, a canonical Majorana mass for the right-handed neutrino is generated, and the light neutrino mass matrix becomes

$$(M^\nu)(M^{\nu^c})^{-1}(M^\nu)^T\, , \tag{4.18}$$

where M^ν is given in eq. (4.9) and M^{ν^c} arises from the coupling $S'\nu^c\nu^c$

$$\mathcal{M}^{\nu^c} = gNa^{\nu^c}a^{\nu^c}A^{s'}B^{\nu^c}B^{\nu^c}\, , \tag{4.19}$$

with

$$A^{s'} = \begin{pmatrix} s'_1 & s'_3\varepsilon_5 & s'_2\varepsilon_5 \\ s'_3\varepsilon_5 & s'_2 & s'_1\varepsilon_5 \\ s'_2\varepsilon_5 & s'_1\varepsilon_5 & s'_3 \end{pmatrix}. \tag{4.20}$$

Note that in eq. (4.18) the mixing B^{ν^c} cancels, so that we may as well assume $a^{\nu^c}B^{\nu^c} = 1$. To present the neutrino mass matrix it is convenient to parameterize the Higgs VEVs as

$$\begin{aligned} v^u_2 &= c_l c_2\, \varepsilon_5^{1/2} v^u_3 \\ v^u_1 &= c_l c_1\, \varepsilon_5^{4/3} v^u_3\, . \end{aligned} \tag{4.21}$$

Inspection of the numerical values of the VEVs in eq. (3.24) shows that the prefactors $c_l \times c_{1,2}$ are of order unity. We will first discuss how to obtain a neutrino mass matrix with maximal mixing and a hierarchy in the mass eigenvalues of order 10, and in the next section will discuss what the appropriate see-saw scale has to be. We will present two examples of neutrino masses with maximal mixing corresponding to the two possible solutions of eq. (4.8), i.e. $\alpha^L, \alpha^{e^c} \sim \varepsilon_5, 1$ or $\alpha^L, \alpha^{e^c} \sim 1, \varepsilon_5$. In the next section we shall see that both possibilities have a see-saw scale of around 10^4 TeV, and generate an effective Higgs μ-term of order 500 TeV.

[8]It is possible that the additional U(1) charges in the string compactification disallow any such Majorana-like coupling with the same field appearing twice. In that case other superpotentials with different assignments are possible, although the phenomenology is somewhat more complicated. We could have for example $SH^uH^d + SS'\nu^c$ or $SH^uH^d + S'S''\nu^c$, in which case the right handed neutrino is mixed with the other singlets, neutral Higgses, winos and binos, in a 14×14 or 17×17 neutralino mass matrix.

First consider choosing $\alpha^L \sim \varepsilon_5$ and $\alpha^{e^c} \sim 1$ so that $a_L \sim 1$ accordingly. In addition we may choose the FI mixing so that the B-matrix for the leptons is permuted with respect to the quarks. We recall that the degeneracy of the v_i^d VEVs allows us to do this without worrying about the charged lepton masses. However, when calculating the mixing angles of the MNS matrix, we must be careful to maintain the correct e, μ, τ generation assignment. (In practice one can take account of this by simply using the relations in eq. (4.21) with the indices permuted, but otherwise leaving the generation labels unchanged.)

The mixing is then given by

$$B^L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_5 \end{pmatrix}. \tag{4.22}$$

In order to get a form of neutrino matrix that gives maximal mixing (with nearly degenerate entries in the 23 submatrix) we allow the low-energy singlet VEVs to have a small hierarchy of their own, given by

$$\begin{aligned} s_1' &= \varepsilon_5^{-3/2} s_3' \\ s_2' &= \varepsilon_5^{-1} s_3' \,. \end{aligned} \tag{4.23}$$

To first order in ε_5 the mass matrix can then be written as proportional to

$$\frac{\varepsilon_5^2 v_3^{u2}}{s_3'} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 0 & c_2 & c_2^2 \end{pmatrix} + O(\varepsilon_5^{5/2}) \,, \tag{4.24}$$

where the 1,2,3 elements correspond to e, μ, τ respectively. The mass hierarchy between second and third generation is now generically $m_{\nu_3}/m_{\nu_2} \approx \sqrt{\varepsilon_5}$ which is of the right order of magnitude. This is because the subleading contributions are suppressed by $\varepsilon_5^{1/2}$ with respect to the leading terms. The first generation mass in this case is negligible, and the other mixing angles (apart from the 23 mixing) can be large, but are sensitive to the precise values of parameters such as c_2, so that it is not possible to get any more predictions in this scheme.

The second possibility has $\alpha^L \approx 1$ and $\alpha^{e^c} \approx \varepsilon_5$ so that $a_L = \varepsilon_1, \varepsilon_3, \varepsilon_1\varepsilon_3$ accordingly. In this case we can use

$$B^L = \begin{pmatrix} \varepsilon_5 & 0 & 0 \\ 0 & 1/\varepsilon_5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.25}$$

and a very mild singlet hierarchy given by

$$a_1^{-1} s_1' = a_3^{-1} s_3' = \varepsilon_5 s_2' \,, \tag{4.26}$$

where a_1 and a_3 are of order unity. The neutrino mass matrix becomes proportional to

$$\frac{v_3^{u2}}{s_3'} \frac{1}{1 - a_1^2 a_3^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (1 - a_1^2 a_3^2) c_2^2 - a_3^2 & 1 \\ 0 & 1 & -a_1^2 \end{pmatrix} + O(\varepsilon_5^{3/2}) \,. \tag{4.27}$$

In this case we again have maximal θ_{23} mixing, but now the neutrino mass hierarchy is determined by how close the constants a_1, a_3 are to unity. Note that the subleading terms are now suppressed by a factor of order $\varepsilon_5^{3/2} \approx 10^{-2}$ which implies that $m_{\nu_1} = 10^{-2} m_{\nu_3} \sim 5 \times 10^{-4}\,\mathrm{eV}$.

5. Absolute values

In the previous sections we were concerned with hierarchies of masses and with mixings, all of which are independent of any overall prefactors in the Yukawas and which therefore depended on only one suppression factor ε_5. We now turn to the absolute values of Yukawas which do depend on $\varepsilon_{1,3}$ and also on additional volume factors.

Having fixed the hierarchies and mixings, we need only concentrate on one mass eigenvalue of any particular particle, which we choose to be, m_c, m_s, m_μ, m_{ν_3}, simply because these are independent of the various α and β factors. As discussed earlier, since the right-handed neutrino coupling is unsuppressed, their values are given by

$$m_c = gNa^{u^c} v_2^u \,, \tag{5.1}$$

$$m_s = gN\varepsilon_1 a^{d^c} v_2^d \,, \tag{5.2}$$

$$m_\mu = gN\varepsilon_3 a^L a^{e^c} v_2^d \,, \tag{5.3}$$

$$m_{\nu_3} = \begin{cases} gN\,(\varepsilon_1\varepsilon_3\varepsilon_5)^2\,(v_3^u)^2/s_3' \\ gN\,\left(\varepsilon_1\varepsilon_3 a^L\right)^2\,(v_3^u)^2/s_3' \,, \end{cases}$$

where the two neutrino masses are for the two possibilities outlined in the previous section. Taking into account eqs. (5.1) and (5.2) above we can write the constraint (3.26) as

$$\frac{67}{(a^{u^c})^2} + \frac{1}{\varepsilon_1^2}\,\frac{0.03}{(a^{d^c})^2} \approx 3 \times 10^4 \,, \tag{5.4}$$

where we have taken $gN \approx 1$. On the other hand, following the discussion below eq. (2.34), we know that $a^{u^c,d^c} \sim \varepsilon_1, \varepsilon_3, \varepsilon_1\varepsilon_3$. Thus, in principle, different combinations between the values of a^{u^c}, a^{d^c} and ε_1 may arise. A possible solution is obtained taking $a^{u^c} \sim a^{d^c} \sim \varepsilon_1 \sim \varepsilon_5 \sim 0.05$, implying $v_2^d \approx 40\,\mathrm{GeV}$ and $v_2^u \approx 27\,\mathrm{GeV}$. For the ratio m_s/m_μ we then find

$$\frac{\varepsilon_1 a^{d^c}}{\varepsilon_3 a^L a^{e^c}} = \frac{m_s}{m_\mu} \,, \tag{5.5}$$

giving

$$\varepsilon_1 \sim \frac{a^L a^{e^c}}{a^{d^c}} \varepsilon_3 \,. \tag{5.6}$$

A solution for this equation is obtained with $a^L a^{e^c} \sim \varepsilon_1 \sim \varepsilon_3$ which is consistent with our earlier requirement that $\alpha^L \alpha^{e^c} \sim \varepsilon_5$. Summarizing, we have obtained the result $\varepsilon_1 \sim \varepsilon_3 \sim \varepsilon_5 \sim \varepsilon \sim 0.05$ implying that $T_1 \sim T_3 \sim T_5 \sim 2$, and in fact, we could find no other choice of a^{u^c,d^c} that would be appropriate. This last observation is extremely interesting, since one would naturally expect T_i moduli VEVs that are dynamically determined by supersymmetry breaking to be of the same order, and here we have found that the fermion masses support that.

Concerning the neutrinos, we found that mildly hierarchical singlet VEVs led to the correct neutrino mass hierarchies and mixings with either of the possibilities $\alpha^L \sim \varepsilon_5$, $\alpha^{e^c} \sim 1$, $a^L \sim 1$ or $\alpha^L \sim 1$, $\alpha^{e^c} \sim \varepsilon_5$, $a^L \sim \varepsilon_1, \varepsilon_3$. In the first case we find a third generation neutrino mass of

$$m_{\nu_3} = gN\varepsilon_1^2\varepsilon_3^2\varepsilon_5^2\frac{v_3^{u2}}{s_3'}\,, \tag{5.7}$$

and taking the ratio of this mass with respect to m_c gives

$$\frac{a^{u^c}}{\varepsilon_1^2\varepsilon_3^2\varepsilon_5^2}\frac{v_2^u s_3'}{(v_3^u)^2} = \frac{m_c}{m_{\nu_3}} \approx 3\times 10^{10}\,, \tag{5.8}$$

which gives us

$$s_3' \approx 10^4\,\mathrm{TeV}\,, \tag{5.9}$$

and therefore, using eq. (4.23),

$$s_2' \approx 2\times 10^5\,\mathrm{TeV}\,, \qquad s_1' \approx 9\times 10^5\,\mathrm{TeV}\,. \tag{5.10}$$

The second neutrino matrix we found gives the same value for s_3' but now we have

$$s_1' \approx s_3' \approx s_2'/20 \approx 10^4\,\mathrm{TeV}\,. \tag{5.11}$$

Summarizing, for the neutrinos we found that large mixing angles, and a hierarchy of ~ 10 are natural if the singlet VEVs are hierarchical. The experimental values of neutrino masses then fixed the singlet VEVs (which in this case are acting as a reduced see-saw scale). The coupling to the Higgs fields, SH^uH^d, is suppressed by a coupling ε so that we may assign an effective μ-term values for each generation, the lightest of which is (assuming $\langle S\rangle \sim \langle S'\rangle$)

$$\mu_3 \approx 500\,\mathrm{TeV}\,. \tag{5.12}$$

It is not clear (without a full minimization of the potential) how this would translate into Higgs masses, however we find it remarkable that they are within reach of the TeV scale required for electroweak symmetry breaking. One possibility that we will not pursue here, is to try to find an assignment of fields which allows the fully suppressed neutrino Dirac masses with $a^L \approx \varepsilon_5^2$. Alternatively one might try to modify the superpotential, with a possible reduction of the μ term through a suppressed $SS'S''$ couplings.

6. Summary of predictions with $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon$

All the results of the previous section were used to restrict the values of $\varepsilon_{1,3,5}$. However it is interesting to take the $\varepsilon_1 \approx \varepsilon_3 \approx \varepsilon_5 = \varepsilon$ condition as a starting principle, in order to summarize our predictions (see footnote 1). First the ratios of quark masses, the 12 element for the CKM matrix and M_W, fixed $\varepsilon \approx 0.05$ and $(v_1^u,\, v_2^u,\, v_3^u) \approx (4.5,\, 27,\, 162)$

GeVand $(v_1^d, v_2^d, v_3^d) \approx (40, 40, 40)$ GeV. It is interesting that for the downs the VEVs are degenerate. We then obtained two successful predictions for the CKM matrix elements;

$$\begin{aligned} (V_{CKM})_{23} &= \frac{1}{\varepsilon}\left(\frac{m_d}{m_b} - \frac{m_u}{m_t}\right) \\ (V_{CKM})_{13} &= \frac{m_s}{m_b} - \frac{m_c}{m_t} + \varepsilon\left(\frac{m_u}{m_c} - \frac{m_d}{m_b}\frac{m_t}{m_c}\right). \end{aligned} \tag{6.1}$$

In the charged lepton sector, we found two successful predictions for mass ratios;

$$\begin{aligned} \frac{m_e}{m_\mu} &= \varepsilon\frac{m_d}{m_s} \\ \frac{m_\mu}{m_\tau} &= \frac{m_s}{m_b}. \end{aligned} \tag{6.2}$$

The absolute values of the quark and charged lepton masses gave us two further predictions;

$$\begin{aligned} m_s &= \varepsilon m_c \\ m_s &= m_\mu . \end{aligned} \tag{6.3}$$

For the neutrinos we found that large mixing angles, and a hierarchy of ~ 10 are natural if the singlet VEVs are mildly hierarchical. The experimental values of neutrino masses then fixed the singlet VEVs (which in this case are acting as a reduced see-saw scale). The coupling to the Higgs fields is suppressed by a coupling of order ε so that we may assign effective μ-term values for each generation, the lightest of which is

$$\mu_3 \sim 500\,\mathrm{TeV}\,. \tag{6.4}$$

It is not clear (without a full minimization of the potential) whether the μ terms here can be further reduced (since the singlet field in the effective μ term is different from that in the Majorana neutrino mass term, as shown in eq. (4.16)). It is also not clear how the mild s' hierarchies would translate into μ-term hierarchies and subsequently into Higgs masses, however we find it intriguing that this rough estimate of scales gave a result that is within reach of the electroweak breaking scale.

7. Conclusions

In this paper we have examined the possibility of generating the fermion mass structure through purely renormalizable couplings in heterotic $\mathbb{Z}_3$ orbifolds with two Wilson lines. The advantages of these models is that they naturally predict three generations, and also that the three generations of Higgs fields give enough freedom to allow an entirely geometric explanation of masses and mixings. In our analysis we found that the Higgs VEVs required only a mild hierarchy in order to fit the experimental values of masses and mixings, and that the mass hierarchies were generated by hierarchical mixing with heavy fields after FI breaking. This is a central feature of the picture presented here.

Our analysis here has been a phenomenological, "bottom-up", one. That is we have assigned the particles to fixed points in a way that can reproduce the experimental data complete with hierarchies. We leave the search for such an assignment to future work but think that it is likely to exist, with issues such as anomaly cancellation (by extra triplets and doublets) being handled by the string construction. In addition we have not completed a full analysis of minimizing the potential along D-flat directions after FI breaking, but have made use of the very general features that such a minimization should have, namely hierarchical mixing of the would-be MSSM fields (i.e. those that couple to the Higgs fields) with other doublet and triplet fields that couple to heavy fields.

In the sense that our analysis is a 'bottom-up' analysis of $\mathbb{Z}_3$ orbifolds, it is intermediate between full string constructions, and models in less restricted extra-dimensional set-ups with fields localized at fixed points or on domain walls (see for example ref. [37]). The consistency conditions of the string constructions provide additional constraints that we think makes this approach more attractive. Conversely our approach may prove to be useful in guiding heterotic orbifold model building. Whilst this paper was in preparation, ref. [34] appeared. In that work a similar geometric set-up was presented for type II models with intersecting D-branes. Although the issue of FI mixing did not arise in that case, the comparative predictiveness is not clear as there are extra 'moduli' associated with the positions of the D-branes.

Acknowledgments

We gratefully acknowledge O. Lebedev for his collaboration during the early stages of this work, mainly in connection with the result of section 3.1. We also thanks S. Khalil for interesting discussions. C. Muñoz is grateful to the members of the IPPP, Durham University, U.K., for their kind hospitality, and also for their support, through the IPPP Visitor Programme, to spend the months of July and August 2002, when most of this work was carried out. The work of S.A. Abel has been supported by a PPARC Opportunity grant. The work of C. Muñoz has been supported in part by the CICYT, under contract FPA2000-0980, and the European Union, under contract HPRN-CT-2000-00148.

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